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# Dirac-type gauge theories and the mass of the Higgs boson

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## Abstract

We discuss the mass of the (physical component of the) Higgs boson in one-loop and top-quark-mass approximation. For this the minimal Standard Model is regarded as a specific (parameterized) gauge theory of Dirac type. It is shown that the latter formulation, in contrast to the usual description of the Standard Model, gives a definite value for the Higgs mass. The predicted value for the Higgs mass depends on the value addressed to the top mass  $m_T$ . We obtain  $m_H = 186 \pm 8$  GeV for  $m_T = 174 \pm 3$  GeV (direct observation of top events), resp.  $m_H = 184 \pm 22$  GeV for  $m_T = 172 \pm 10$  GeV (Standard Model electroweak fit). Although the Higgs mass is predicted to be near the upper bound,  $m_H$  is in full agreement with the range  $114 \leq m_H < 193$  GeV that is allowed by the Standard Model. We show that the inclusion of (Dirac) massive neutrinos does not alter the results presented. We also briefly discuss how the derived mass values are related to those obtained within the frame of non-commutative geometry.

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## 1. Introduction

The Higgs boson (more precisely, the physical component thereof) is known to be the last outstanding particle predicted by the (minimal) Standard Model (STM). Within the usual description of the STM the expected mass range of the Higgs boson is restricted to the interval [114, 193] GeV (cf [Ros03]). This prediction of the range of the Higgs mass results from including quantum corrections and additional experimental input. Of course, over the last

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decade there have been many attempts to better specify the value of this mass range using different mathematical approaches to the STM. One particular mathematical approach worth mentioning here is given within the realm of non-commutative geometry; see for instance [Coq89, Ka91, Ka92, Ka93, GBV93, Con94, KS96] and more recently [CCM06]. For similar approaches one may consult, for instance, [HPS91, MO94, MO96] and the appropriate references therein.

In this paper we discuss the Higgs mass within the STM using the geometrical frame of (parameterized) Dirac-type gauge theories (GTDT). The mathematical background of GTDT is discussed in some detail in [TT05, Tol07]. However, in order to be self-contained we present a purely local description of GTDT which is also needed to derive the relations for the Higgs mass.

The basic idea of GTDT is to introduce a general geometrical setup to describe (a certain class of) gauge theories in terms of fermions. Hence, the fundamental ingredients of GTDT are so-called ‘generalized Dirac operators’. They basically differ from the usual Dirac operator by a general zero-order term. This zero-order term in turn may be used to define a gauge potential not only by a single 1-form but also by forms of various degrees (please, see below). In this sense, Dirac-type operators may be regarded as more general than connections. Physically speaking, generalized Dirac operators permit incorporation of different fields into one single mathematical object, which in turn are physically motivated by the postulated interactions of the fermions considered. Another advantage of describing gauge theories in terms of generalized Dirac operators is that the latter naturally induce specific Lagrangian densities. These densities can be shown to be equivariant with respect to the action of the full gauge group including the gauge group of Yang–Mills, of gravity and the diffeomorphism group of the underlying (spacetime) manifold. In this sense one may say that the gauge theories defined by the corresponding Lagrangians have a ‘square root’ in terms of generalized Dirac operators. This is not only conceptually more satisfying than the usual ‘adding of actions’, but may also have some phenomenological consequences. Accordingly, the present paper aims at showing how the geometrical setup of GTDT allows the specification of the range of the Higgs mass of the STM. The calculations presented are similar to those given, for instance, in [KS97]. In particular, we restrict our discussion to one-loop-order and top-quark-mass approximation.

One basic feature of GTDT is that it is logically inconsistent to assume that spacetime is flat. This is because a Dirac-type operator generically yields a non-vanishing energy–momentum tensor which in turn implies a non-vanishing curvature of spacetime. However, one may still assume that gravitational effects may be negligible in comparison with some given energy scale naturally implied by the gauge theory at hand. In fact, this is our basic assumption as far as the presented calculations of the Higgs mass are concerned (for a corresponding justification see also the concluding remarks related to this issue). Moreover, it is assumed that the Standard Model (as well as perturbation theory) is valid up to a certain energy scale  $E_c$ , which is much smaller than the Planck scale but significantly higher than the scale set by the yet to find Higgs mass.

The paper is organized as follows. In section 2 we present a purely local description of GTDT as is needed to follow the line of reasoning involved in the calculation of the Higgs mass. In section 3 we summarize the STM as it is described as a special GTDT. There we also present a natural parametrization of the general mathematical scheme that is presented in [TT05]. In section 4 we discuss the parameter relations between the appropriately parameterized GTDT of the STM with its usual mathematical description. We then discuss the resulting renormalization flow equations for the energy dependence of the coupling constants to one-loop-order and in top-quark-mass approximation. This is done in the  $\overline{MS}$  scheme. Afterward we discuss the possible changes when a massive neutrino sector is included. We

also discuss in this section the principal bounds of the Higgs mass within GTDT. In section 5 we compare our results with those presented within the geometrical scheme of non-commutative geometry. We conclude with some comments on the results discussed in this paper. In the appendix we briefly summarize the relations between the gauge couplings and the empirical parameters used in this paper.

## 2. GTDT—a local description

In this section, we present a local description of gauge theories of Dirac type in the case of a four-dimensional (parallelizable) Lorentzian manifold. This description will then be applied to the (minimal) Standard Model in the following section in order to obtain some statements about the mass of the (physical component of the) Higgs boson.

Basically, a GTDT is given by the following universal (Dirac-) Lagrangian:

$$\mathcal{L}_D := (\bar{\psi} iD\psi + V_D)\sqrt{-|g|} d^4x, \quad (1)$$

$$V_D \equiv \frac{N}{2}r_M + \text{tr}(\gamma^{\mu\nu}[\theta_\mu, \theta_\nu]) + \frac{1}{8}g_{\mu\nu} \text{tr}(\gamma^\sigma[\theta_\sigma, \gamma^\mu]\gamma^\lambda[\theta_\lambda, \gamma^\nu]). \quad (2)$$

Here,  $|g| \equiv \det(g_{\mu\nu})$  and  $r_M$  denotes the Ricci scalar curvature with respect to the Lorentz metric  $g_{\mu\nu}$  of signature  $-2$ . The Dirac matrices  $\gamma^\mu \in \mathbf{M}_N(\mathbb{C})$  fulfil the Clifford relation  $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}\mathbf{1}_N$ , with  $g^{\mu\nu}$  being the inverse of  $g_{\mu\nu}$ . Also, we use the common abbreviation  $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$  for the generators of the (proper orthochronous) Lorentz transformations in the spin representation. The dimension  $N \equiv 4N_F$  of the representation space is given by the ‘fermion representation’, i.e.  $\psi \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{C}^4 \otimes \mathbb{C}^{N_F})$ . In the following  $\mathcal{M} \subset \mathbb{R}^4$  denotes some open (connected) subset such that  $T\mathcal{M} \simeq \mathcal{M} \times \mathbb{R}^4$ .

The Dirac operator  $D = \gamma^\mu \nabla_\mu^D$  is defined in terms of the (Dirac) connection

$$\begin{aligned} \nabla_\mu^D &:= \partial_\mu + \omega_\mu + \theta_\mu \\ &\equiv \nabla_\mu^S + \theta_\mu, \end{aligned} \quad (3)$$

with  $\omega_\mu \equiv \frac{1}{4}\gamma^{\alpha\beta}\omega_{\mu\alpha\beta} \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_N(\mathbb{C}))$  being the spin-connection form and  $\nabla_\mu^S$  the corresponding spin connection with respect to  $g_{\mu\nu}$ . Also, the 1-form  $\theta_\mu \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_N(\mathbb{C}))$  denotes a general gauge potential. The connection (3) is called a *Clifford connection* (or, twisted spin-connection) if the general gauge potential  $\theta_\mu$  fulfils

$$[\theta_\mu, \gamma^\nu] = 0. \quad (4)$$

In this case we write  $\theta_\mu = A_\mu$ , such that a Clifford connection reads<sup>4</sup>

$$\begin{aligned} \nabla_\mu^D &\equiv \nabla_\mu^{\text{Cl}} := \partial_\mu + \omega_\mu + A_\mu \\ &= \nabla_\mu^S + A_\mu. \end{aligned} \quad (5)$$

Accordingly, the Dirac operator  $D$  is then called a ‘twisted spin Dirac operator’. However, for a general gauge potential  $\theta_\mu$  one has  $[\theta_\mu, \gamma^\nu] \neq 0$ . In this more general situation the appropriate Dirac operator  $D$  is known as a *generalized Dirac operator* (or, operator of Dirac type), see for example in [ABS64, BGV96]. In what follows, however, we will refer to  $D = \gamma^\mu \nabla_\mu^D$  simply as a Dirac operator, even in the case where  $D$  is defined with respect to general  $(g_{\mu\nu}, \theta_\mu)$ .

Of course, a general gauge potential  $\theta_\mu$  can be decomposed as  $\theta_\mu = A_\mu + (\theta_\mu - A_\mu) \equiv A_\mu + H_\mu$ . It can be shown that the *Dirac potential* (2) is independent of such a decomposition.

<sup>4</sup> In [TT05] the covariant derivative of a Clifford connection is denoted by  $\partial_A$ . It should not be confounded with the lifted Levi-Civita connection.

Hence, without loss of generality we may decompose a general connection (3) into a Clifford connection plus a general gauge potential

$$\begin{aligned} \nabla_\mu^D &= \nabla_\mu^S + A_\mu + H_\mu \\ &= \nabla_\mu^{Cl} + H_\mu \end{aligned} \tag{6}$$

and thereby substitute  $\theta_\mu$  by  $H_\mu$  in (2). This general gauge potential  $H_\mu$  can be expressed also in terms of the Dirac operator  $D$  itself:

$$\begin{aligned} H_\mu &= -\frac{1}{4}g_{\mu\nu}\gamma^\nu(D - \gamma^\sigma\nabla_\sigma^{Cl}) \\ &\equiv \xi_\mu\Phi_D. \end{aligned} \tag{7}$$

Note that  $\gamma^\mu\xi_\mu = -\frac{1}{4}g_{\mu\nu}\gamma^\mu\gamma^\nu = \mathbf{1}_N$  and  $\Phi_D = \gamma^\mu H_\mu$ . Thus, we may decompose any (generalized) Dirac operator as

$$D = \gamma^\sigma\nabla_\sigma^{Cl} + \Phi_D. \tag{8}$$

Note that for any gauge potential  $H'_\mu$  which fulfils the two requirements:  $\gamma^\mu H'_\mu = \gamma^\mu H_\mu$  and  $\xi_\mu\gamma^\nu H'_\nu = H'_\mu$ , one infers that  $H'_\mu = H_\mu$ . Also note that

$$\Phi_D = \sum_{k=0}^4 \sum_{0 \leq v_1 < \dots < v_k \leq 3} \gamma^{v_1} \dots \gamma^{v_k} \chi_{v_1 \dots v_k}^{(k)}, \tag{9}$$

with  $[\chi_{v_1 \dots v_k}^{(k)}, \gamma^\mu] = 0$  being considered as  $k$ -forms on  $\mathcal{M}$  which take values in  $\mathbf{M}_N(\mathbb{C})$ . The lowest order contribution  $\chi^{(0)}$  is characterized by  $[\Phi_D, \gamma^\mu] = 0$ . In contrast, the highest order contribution

$$\sum_{0 \leq v_1 < \dots < v_4 \leq 3} \gamma^{v_1} \dots \gamma^{v_4} \chi_{v_1 \dots v_4}^{(4)} = \gamma_5 \phi \tag{10}$$

fulfils the condition

$$\{\Phi_D, \gamma^\mu\} = 0, \tag{11}$$

with  $\phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{N_F}(\mathbb{C}))$  and  $\gamma_5 = i\gamma^0 \dots \gamma^3$  the canonical grading operator on the spinor space (such that  $\mathbb{C}^4 = S_L \oplus S_R$  decomposes into the ‘left-handed’ and ‘right-handed’ spinors). Condition (11) is analogous to (4) for it implies

$$[\theta_\mu, \gamma^\nu] = -\frac{1}{2}\delta_\mu^\nu \gamma^\lambda \theta_\lambda. \tag{12}$$

Moreover, the first-order contribution only yields a re-definition of the Yang–Mills gauge potential  $A_\mu$ . Hence, in the following we will omit the first-order part in  $\Phi_D$ .

The *relative curvature* of a Dirac-type operator is defined as

$$\begin{aligned} F_{\mu\nu}^\theta &:= \nabla_\mu^S \theta_\nu - \nabla_\nu^S \theta_\mu + [\theta_\mu, \theta_\nu] \\ &= \partial_\mu \theta_\nu - \partial_\nu \theta_\mu + [\theta_\mu, \theta_\nu] + [\omega_\mu, \theta_\nu] - [\omega_\nu, \theta_\mu]. \end{aligned} \tag{13}$$

It naturally decomposes as

$$\begin{aligned} F_{\mu\nu}^\theta &= F_{\mu\nu}^A + F_{\mu\nu}^{A,H} \\ &= F_{\mu\nu}^A + F_{\mu\nu}^H + \kappa_{\mu\nu}^{A,H}. \end{aligned} \tag{14}$$

Here,  $\kappa_{\mu\nu}^{A,H} := [A_\mu, H_\nu] - [A_\nu, H_\mu]$  abbreviates the ‘interaction term’ between the gauge potentials  $A_\mu$  and  $H_\mu$ . The curvature

$$\begin{aligned} F_{\mu\nu}^{A,H} &:= F_{\mu\nu}^D - F_{\mu\nu}^{Cl} \\ &= \nabla_\mu^{Cl} H_\nu - \nabla_\nu^{Cl} H_\mu + [H_\mu, H_\nu] \end{aligned} \tag{15}$$

denotes the relative curvature of  $H_\mu$  with respect to  $(\omega_\mu, A_\mu)$  and  $F_{\mu\nu}^D$  is the curvature with respect to the Dirac connection (3). In the case of  $H_\mu = 0$  (i.e.  $\theta_\mu = A_\mu$ ) the relative curvature is called the ‘twisting curvature’ of  $D$ . Since  $[\omega_\mu, A_\nu] = 0$ , the twisting curvature  $F_{\mu\nu}^A = \nabla_\mu^S A_\nu - \nabla_\nu^S A_\mu + [A_\mu, A_\nu]$  coincides with the usual Yang–Mills field strength, provided Clifford connections  $\nabla_\mu^{\text{Cl}}$  are identified with Yang–Mills connections  $\nabla_\mu^A \equiv \partial_\mu + A_\mu$ .

There is a distinguished class of Dirac-type operators called *Dirac operators of Yukawa type* (cf [TT05]). These operators are defined by

$$\Phi_D := \gamma_5 \phi. \quad (16)$$

Note that  $D$  is odd if and only if  $\mathbb{C}^{N_F} = E_L \oplus E_R$  and  $\phi$  is odd. Assuming that  $\phi \neq 0$ , it can be shown that the field equations for Dirac operators of Yukawa type give rise to the existence of a constant (skew-hermitian) matrix function  $\mathcal{D} \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{M}_{N_F}(\mathbb{C}))$  and a real-valued smooth function  $h \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$  such that

$$\phi = h\mathcal{D}. \quad (17)$$

This reduces the gauge symmetry group to the isotropy group of  $\mathcal{D}$ . Accordingly, a Yukawa-type Dirac operator is said to be in the *unitary gauge* if it reads

$$D = \gamma^\sigma \nabla_\sigma^{\text{Cl}} + \gamma_5 \mathcal{D}. \quad (18)$$

A Yukawa-type Dirac operator is said to represent a *fermionic vacuum* if

$$\begin{aligned} D &= \gamma^\mu (\partial_\mu + \omega_\mu) + \gamma_5 \mathcal{D} \\ &\equiv \gamma^\mu \nabla_\mu^S + \gamma_5 \mathcal{D} \end{aligned} \quad (19)$$

and  $g_{\mu\nu}$  fulfils the Einstein equation

$$R_{\mu\nu} = \kappa_{\text{gr}} \text{tr } \mathcal{D}^2 g_{\mu\nu}. \quad (20)$$

A general Yukawa-type operator is then considered as a perturbation of a fermionic vacuum. Note that with respect to the latter any Yukawa-type operator corresponds to  $(h, A_\mu, h_{\mu\nu})$ . Here, the metric  $h_{\mu\nu}$  is considered as a perturbation of  $g_{\mu\nu}$  which satisfies the Einstein equation with the energy–momentum tensor being defined with respect to  $(\psi, h, A_\mu)$ . As already mentioned in the introduction, we will neglect the influence of a non-flat spacetime and assume that  $g_{\mu\nu} = h_{\mu\nu} \approx \eta_{\mu\nu}$ . Some appropriate comments on a justification of this assumption will be given in the conclusion.

To lowest order the metric  $g_{\mu\nu}$  is fully determined by the spectrum of  $\mathcal{D}^2$ . In contrast, the field equations of a Yukawa-type operator do not determine either the Yang–Mills connection (i.e. the gauge potential  $A_\mu$ ), or the (physical component of the) Higgs field  $h$ . For this one has to slightly enlarge the class of Yukawa-type operators, which is referred to as *Pauli–Yukawa-type Dirac operators* (PDY). They are defined by Dirac operators of the form

$$\begin{aligned} D &= \begin{pmatrix} \gamma^\mu (\nabla_\mu^S + \theta_\mu) & -\frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^\theta \\ \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^\theta & \gamma^\mu (\nabla_\mu^S + \theta_\mu) \end{pmatrix} \\ &\equiv \gamma^\mu (\nabla_\mu^S + \theta_\mu) + \mathcal{I} \left( \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^\theta \right), \end{aligned} \quad (21)$$

with the fermion representation space being doubled. Here, the *Higgs gauge potential* reads  $H_\mu := \xi_\mu \gamma_5 \phi$  and thus  $\theta_\mu = A_\mu + \xi_\mu \gamma_5 \phi$ . Moreover,  $\mathcal{I} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  may be regarded as defining an additional complex structure for  $\mathcal{I}^2 = -\mathbf{1}_2$ . Hence, in the case of a twisted spin Dirac operator (i.e.  $\theta_\mu = A_\mu$ ) a PDY reduces to  $D = \gamma^\mu \nabla_\mu^{\text{Cl}} + \mathcal{I} \left( \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu}^A \right)$ , where the second part formally looks like the well-known ‘Pauli term’.

The relative curvature  $F_{\mu\nu}^{A,H}$  of the Higgs gauge potential explicitly reads

$$F_{\mu\nu}^{A,H} = \gamma_5 (\xi_\mu [\nabla_\nu^A, \phi] - \xi_\nu [\nabla_\mu^A, \phi]) + [\xi_\mu, \xi_\nu] \phi^2. \quad (22)$$

Note that this (relative) curvature depends on the total field content  $(g_{\mu\nu}, \phi, A_\mu)$ .

With respect to a fermionic vacuum (19) one may consider Clifford connections which are compatible with the vacuum, i.e. gauge potentials  $A_\mu$  which satisfy

$$[A_\mu, \mathcal{D}] = 0. \tag{23}$$

In this case the interaction term  $\kappa_{\mu\nu}^{A,H}$  vanishes identically and the relative curvature (14) reduces to

$$F_{\mu\nu}^\theta = F_{\mu\nu}^A + F_{\mu\nu}^H. \tag{24}$$

In fact, the latter relation is equivalent to the compatibility of a Clifford connection with a fermionic vacuum.

The interaction term  $\kappa_{\mu\nu}^{A,H}$  has a simple physical meaning. With respect to the fermionic vacuum it corresponds to the ‘Yang–Mills mass matrix’. Indeed, the ‘generalized Yang–Mills Lagrangian’  $\text{tr} F_{\mu\nu}^\theta F_{\mu\nu}^{\mu\nu}$  clearly yields a term like

$$\begin{aligned} g^{\alpha\beta} g^{\mu\nu} \text{tr} \kappa_{\mu\alpha}^{A,H} \kappa_{\nu\beta}^{A,H} &\sim (1+h)^2 g^{\mu\nu} \text{tr}(\mathcal{D}\{A_\mu, A_\nu\}\mathcal{D}) \\ &\sim (1+h)^2 M_{ab}^2 g^{\mu\nu} A_\mu^a A_\nu^b, \end{aligned} \tag{25}$$

with  $M_{ab}^2 := \text{tr}(\mathcal{D}\{T_a, T_b\}\mathcal{D})$  being proportional to the (squared) Yang–Mills mass matrix and  $A_\mu = A_\mu^a T_a$ . Note that the generators  $T_a \in \mathbf{M}_{N_F}(\mathbb{C})$  refer to the fermion representation. Moreover,  $M_{ab}^2$  equals zero exactly for those generators which commute with  $\mathcal{D}$ .

We emphasize that the ‘generalized Pauli term’  $\gamma^{\mu\nu} F_{\mu\nu}^\theta$  in (21) does not contribute to the fermionic part in (1) when restricted to the real sub-space of ‘particles-anti-particles’. It only contributes to the Dirac potential  $V_D$ . More precisely, for  $D$  of Pauli–Yukawa-type one obtains the total Lagrangian

$$\mathcal{L}_D = (2\bar{\psi}(i\gamma^\mu \nabla_\mu^{\text{Cl}} + i\gamma_5\phi)\psi + V_D)\sqrt{-|g|} d^4x, \tag{26}$$

$$V_D = \lambda_{\text{gr}} r_M - (\lambda_{\text{YM}} \text{tr} F_{\mu\nu}^A F_{\mu\nu}^{A\dagger} + \lambda_H \text{tr} \nabla_\mu \phi^\dagger \nabla^\mu \phi - V_H). \tag{27}$$

Here,  $\nabla_\mu \phi \equiv [\nabla_\mu^A, \phi] = \partial_\mu \phi + [A_\mu, \phi]$  and

$$V_H = \alpha_H (\text{tr} \phi^\dagger \phi)^2 - \beta_H \text{tr} \phi^\dagger \phi \tag{28}$$

is the usual *Higgs potential* of the (minimal) Standard Model and  $\lambda_{\text{gr}}, \lambda_{\text{YM}}, \lambda_H, \alpha_H, \beta_H$  are real parameters to be specified in the following section. For a more detailed discussion, in particular, of the occurrence of the factor 2 and of the grading involution  $\gamma_5$  in the Yukawa coupling and the geometrical meaning of (25), we again refer to [TT05].

In the following section we will make use of (26) and (27) which formally looks like the total Lagrangian of the (minimal) Standard Model including gravity. In fact, for specific Yukawa-type Dirac operators one may appropriately re-write the Dirac–Lagrangian (26) to get exactly the form of the STM-Lagrangian. The scheme proposed may also have some phenomenological consequences since (26) is derived in one stroke by a specific class of Pauli–Yukawa-type operators. For this, however, one still has to take into account the different mass dimensions of the various fields and to also include an appropriate parametrization of both the general Dirac–Lagrangian (1) and the specific class of Dirac operators one deals with. Of course, the parametrization cannot be arbitrary. It has to be compatible with the geometrical setup. Basically, the motivation of an appropriate parametrization of the geometrical scheme considered comes from physics and is analogous to the introduction of the gauge-coupling constants<sup>5</sup> in ordinary (non-Abelian) Yang–Mills gauge theories (please, see below). Clearly,

<sup>5</sup> These should not being regarded as a generalization of the electric charge, for the mathematical origin of the latter is quite different from the gauge-coupling constants.

the possible phenomenological implications of a specific geometrical description, for example, of the STM, strongly depend on the parametrization of the geometrical scheme considered (cf our discussion in section 5).

### 3. The (minimal) STM as a GTDT

As discussed in the previous section the STM-Lagrangian has a natural ‘square root’ in terms of Pauli–Yukawa-type Dirac operators (PDY). This holds true in particular for the Higgs sector. However, within the framework of Dirac-type gauge theories the Higgs field  $\phi$  transforms with respect to the full fermion representation of the gauge group. This is in contrast to the minimal Standard Model where the Higgs field is supposed to transform with respect to a specific sub-representation of the fermion representation  $\rho_F$  (see below). These two representations are related to the Yukawa-coupling matrix  $G_Y$  which can be considered as a linear mapping from the representation space of the Higgs field to the representation space of left-handed fermions. For a general discussion we again refer to [TT05]. In what follows, we will restrict ourselves to the specific case of the minimal Standard Model (see, for instance, [Nac90]).

#### 3.1. Data of the (minimal) STM as a specific GTDT

To specify a GTDT one has to choose a gauge group  $G$ , a unitary representation  $\rho_F$  thereof, as well as some (class of) Dirac operators  $D$ . In the case of the (minimal) STM these data are specified by (here, we adopt the same notation as was used in [Tol98]):

- $G$  equals  $SU(3) \times SU(2) \times U(1)$ .
- $\rho_F$  equals the fermion representation

$$\begin{aligned} E_L &:= \bigoplus_1^3 [(1, 2, -1/2) \oplus (3, 2, 1/6)], \\ E_R &:= \bigoplus_1^3 [(1, 1, -1) \oplus (3, 1, -1/3) \oplus (3, 1, 2/3)], \end{aligned} \quad (29)$$

where  $(n_3, n_2, n_1)$  denote the tensor product, respectively, of an  $n_3$ -dimensional representation of  $SU(3)$ , an  $n_2$ -dimensional representation of  $SU(2)$  and a one-dimensional representation of  $U(1)$  with ‘hypercharge’  $y$ :  $\rho(e^{i\theta}) := e^{iy\theta}$ ,  $y \in \mathbb{Q}$ ,  $\theta \in [0, 2\pi[$ .

More explicitly, we have

$$\rho_F := \rho_L \oplus \rho_R : SU(3) \times SU(2) \times U(1) \rightarrow \text{Aut}(E) \subset U(45) \quad (30)$$

with

$$\rho_L(\mathbf{c}, \mathbf{w}, b) := \begin{pmatrix} \mathbf{c} \otimes \mathbf{1}_N \otimes \mathbf{w}b_L^q & 0 \\ 0 & \mathbf{1}_N \otimes \mathbf{w}b_L^l \end{pmatrix}, \quad (31)$$

$$\rho_R(\mathbf{c}, \mathbf{w}, b) := \begin{pmatrix} \mathbf{c} \otimes \mathbf{1}_N \otimes \mathbf{B}_R^q & 0 \\ 0 & \mathbf{B}_R^l \end{pmatrix} \quad (32)$$

and

$$\begin{aligned} E &\equiv E_L \oplus E_R \\ &\simeq [(\mathbb{C}_q^{18} \oplus \mathbb{C}_l^6)]_L \oplus [(\mathbb{C}^9 \oplus \mathbb{C}^9)_q \oplus \mathbb{C}_l^3]_R. \end{aligned} \quad (33)$$



- The Dirac operator  $D$  is of Pauli–Yukawa-type with Yukawa coupling  $\phi$  given by

$$\phi \equiv i \begin{pmatrix} 0 & \tilde{\phi} \\ \tilde{\phi}^\dagger & 0 \end{pmatrix} \tag{34}$$

with

$$\begin{aligned} \tilde{\phi} \equiv G_Y(\varphi) &:= \begin{pmatrix} \mathbf{1}_3 \otimes (\mathbf{g}^q \otimes \varphi, \mathbf{g}^q \otimes \epsilon \bar{\varphi}) & 0 \\ 0 & \mathbf{g}^l \otimes \varphi \end{pmatrix} \\ &\equiv \begin{pmatrix} \mathbf{1}_3 \otimes \tilde{\varphi}_q & 0 \\ 0 & \tilde{\varphi}_l \end{pmatrix}. \end{aligned} \tag{35}$$

Here, respectively,  $\mathbf{g}^q, \mathbf{g}^l \in \mathbf{M}_N(\mathbb{C})$  denote the matrices of the Yukawa-coupling constants for quarks of electrical charge  $-1/3$  and  $2/3$  (i.e. of quarks of ‘d’-type, and of ‘u’-type) and  $\mathbf{g}^l \in \mathbf{M}_N(\mathbb{C})$  is the matrix of the Yukawa-coupling constants for the leptons of charge  $-1$  (i.e. of leptons of ‘electron’ type). While  $\mathbf{g}^q$  and  $\mathbf{g}^l$  can be assumed to be diagonal and real, the matrix  $\mathbf{g}^q$  is related to the Kobayashi–Maskawa matrix and therefore is neither diagonal nor real. The ‘weak hyper-charges’ for the left- and right-handed quarks (indicated by the superscript ‘q’) and leptons (superscript ‘l’) are defined by  $\rho(b) := e^{iy\theta}$ ,  $b \in U(1)$ ,  $y \in \mathbb{Q}$ ,  $\theta \in [0, 2\pi[$ . Then, the two by two diagonal matrices  $\mathbf{B}_R^q$  and  $\mathbf{B}_R^l$  in (32) are  $\mathbf{B}_R^q := \text{diag}(b_R^{d'}, b_R^u)$  and  $\mathbf{B}_R^l := b_R^l \mathbf{1}_N$ . Here,  $b_R^u := e^{iy_R^u \theta}$ , with  $y_R^u$  being the hypercharge of the right-handed quarks of ‘u-type’ and similar for the other quarks and leptons.

In (35),  $\varphi \in C^\infty(\mathcal{M}, \mathbb{C}^2)$  denotes the complex Higgs field of the minimal Standard Model. It carries the specific sub-representation  $\rho_H$  of the fermion representation  $\rho_F$  :

$$\begin{aligned} \rho_H : SU(3) \times SU(2) \times U(1) &\longrightarrow U(2) \\ (\mathbf{c}, \mathbf{w}, b) &\mapsto \mathbf{w} e^{iy_h \theta}. \end{aligned} \tag{36}$$

Finally,  $\epsilon$  is the anti-diagonal matrix  $\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , which intertwines the fundamental representation of  $SU(2)$  and its conjugate complex representation, and  $\bar{\varphi}$  here means the complex conjugate of  $\varphi$ .

As a sub-representation of  $\rho_F$  the representation  $\rho_H$  is fixed by the relations of the hyper-charges of the quarks and the leptons:

$$y_h = y_L^l - y_R^l = y_L^q - y_R^{d'} = y_R^u - y_L^q. \tag{37}$$

Here,

$$(y_L^q, y_L^l) = (1/6, -1/2), \quad ((y_R^{d'}, y_R^u), y_R^l) = ((-1/3, 2/3), -1), \tag{38}$$

according to the fermion representation (29).

The corresponding fermionic vacuum  $\mathcal{D}$  is given by

$$\mathcal{D} := i \begin{pmatrix} 0 & \mathbf{M} \\ \mathbf{M}^\dagger & 0 \end{pmatrix}, \tag{39}$$

with

$$\mathbf{M} \equiv \begin{pmatrix} \mathbf{1}_3 \otimes \mathbf{M}_q & 0 \\ 0 & \mathbf{M}_l \end{pmatrix}, \quad \mathbf{M}_q \equiv \begin{pmatrix} 0 & \mathbf{m}^{d'} \\ \mathbf{m}^u & 0 \end{pmatrix}, \quad \mathbf{M}_l \equiv \begin{pmatrix} 0 \\ \mathbf{m}^l \end{pmatrix}, \tag{40}$$

where, respectively, the matrices  $\mathbf{m}^l := \frac{v}{\sqrt{2}} \mathbf{g}^l \in \mathbf{M}_N(\mathbb{C})$  and  $\mathbf{m}^u := \frac{v}{\sqrt{2}} \mathbf{g}^q \in \mathbf{M}_N(\mathbb{C})$  denote the ‘mass matrices’ of the charged leptons (l) and quarks (q) of ‘u-type’. They can be assumed to be diagonal and real. The corresponding  $N \times N$  matrix  $\mathbf{m}^{d'} := \frac{v}{\sqrt{2}} \mathbf{g}^{q'}$  of

‘d-type’ quarks is neither diagonal nor real. It is related to the mass matrix of ‘d-type’ quarks  $\mathbf{m}^d = \text{diag}(m^{d_1}, \dots, m^{d_N})$ ,  $m^{d_k} \in \mathbb{R}$ , via the Kobayashi–Maskawa matrix  $\mathbf{V} \in \text{U}(N)$ :  $\mathbf{m}^{d'} = \mathbf{V}\mathbf{m}^d\mathbf{V}^*$ . Here and above  $N$  is of appropriate size that is defined by the fermion representation (29) (or (33)) and  $v > 0$  is the ‘vacuum expectation value’ of the Higgs boson.

With the choice of  $(G, \rho_F, D)$  we have mathematically specified a particular Dirac-type gauge theory. However, from a physical perspective we still have to appropriately parameterize this GTDT. Of course, the parametrization cannot be arbitrary. It should be in accordance with the geometrical frame of a GTDT. As we mentioned already, there are basically two objects which can be parameterized: the Dirac–Lagrangian (1) and the general Dirac operator defined by (3). Note that in the specific case at hand the parameters introduced by the Yukawa coupling matrices only arise because of the change of the representation of the Higgs field (i.e. to consider  $\phi$  as a function of  $\varphi$ ).

### 3.2. Geometrical parametrization

In general, an admissible parametrization of (1) is given by the commutant of the fermion representation  $\rho_F$ . This is similar to the introduction of the Yang–Mills-coupling constant for each simple gauge group. To explain this let us consider the case where  $G = \text{SU}(2)$ ,  $\rho_F$  is some unitary representation thereof and  $D$  is specified by (21) with  $\theta_\mu = A_\mu$ . In this case, the Dirac–Lagrangian (26) reduces to

$$\mathcal{L}_D = (2\bar{\psi} i\gamma^\mu \nabla_\mu^{\text{Cl}} \psi + V_D) \sqrt{-|g|} d^4x, \quad (41)$$

$$V_D = \lambda_{\text{gr}} r_M - \lambda_{\text{YM}} \text{tr} F_{\mu\nu}^A \dagger F_A^{\mu\nu}. \quad (42)$$

Here, the constants  $\lambda_{\text{gr}}$ ,  $\lambda_{\text{YM}}$  are purely numerical and basically fixed by the dimension of spacetime and the dimension of the fermion representation. To get started we may re-scale  $\psi$  to get rid of the factor 2 in the fermionic part of the Dirac–Lagrangian. Moreover, we may introduce a relative constant  $\lambda_D \in \mathbb{R}$  and thereby replace the total Lagrangian (41) by

$$\mathcal{L}_D = (\bar{\psi} i\gamma^\mu \nabla_\mu^{\text{Cl}} \psi + \lambda_D V_D) \sqrt{-|g|} d^4x. \quad (43)$$

Of course, the free parameter  $\lambda_D$  can be absorbed by  $\lambda_{\text{gr}}$ ,  $\lambda_{\text{YM}}$ , which will then be treated as arbitrary free parameters. In particular,  $\lambda_{\text{gr}}$  will be proportional to Newton’s gravitational constant (or the inverse square of the Planck length  $\ell_P$ ) after we have introduced an appropriate length-scale to give the various fields their correct physical dimensions (see below). Next, we take into account that all fields are represented in the fermion representation  $\rho_F$ . Accordingly, there is another numerical constant  $\lambda_F$ , which only depends on  $\rho_F$ , such that we may re-write the Yang–Mills part in (42) as

$$\text{tr} F_{\mu\nu}^A \dagger F_A^{\mu\nu} = \lambda_F \langle F_{\mu\nu}^A, F_A^{\mu\nu} \rangle \quad (44)$$

Here,  $\langle \cdot, \cdot \rangle$  denotes the Killing form on  $\mathfrak{su}(2)$ . Since the latter is proportional to the usual trace with respect to the fundamental representation one may re-write again the Yang–Mills part as

$$\lambda_{\text{YM}} \text{tr} F_{\mu\nu}^A \dagger F_A^{\mu\nu} = \frac{1}{2g_{\text{YM}}^2} \text{tr} F_{\mu\nu}^A F_A^{\mu\nu} \quad (45)$$

where on the right-hand side  $F_{\mu\nu}^A$  is supposed to be in the fundamental representation of  $\mathfrak{su}(2)$ . It follows that

$$\lambda_{\text{YM}} = \lambda_{\text{rep}} / g_{\text{YM}}^2 \quad (46)$$

with  $\lambda_{\text{rep}}$  being a numerical constant which is basically fixed by the fermion representation and the fundamental representation. The constant  $g_{\text{YM}}$  denotes the usual Yang–Mills-coupling constant which parameterizes the most general Killing form on  $\text{Lie}(G)$ .

The specific example discussed so far can be slightly generalized by taking into account the reducibility of  $\rho_F$ . Let  $\mathfrak{Z} = \mathfrak{Z}^\dagger$  be the most general element of the corresponding commutant. In fact,  $\mathfrak{Z}$  can be regarded as a constant mapping commuting with the action of the Clifford algebra, which can be expressed in terms of the Dirac operator as follows:

$$[D, \mathfrak{Z}] = 0. \tag{47}$$

Note that, when considered as a condition on the mapping  $\mathfrak{Z}$ , (47) cannot be weakened either by the condition  $[D^k, \mathfrak{Z}] = 0$  for some integer  $k > 1$ , or by  $\{D^k, \mathfrak{Z}\} = 0$  for some integer  $k \geq 1$ . Indeed, the former case reduces to condition (47) and the anti-commutator condition yields  $\mathfrak{Z} = 0$ . Basically, the reason is that only the operator  $[D, \mathfrak{Z}]$  acts as a zero-order operator on sections  $\psi$ .

Hence, we may generalize the Yang–Mills part in the Dirac–Lagrangian (41) by

$$\text{tr}(\mathfrak{Z} F_{\mu\nu}^A F_A^{\mu\nu}). \tag{48}$$

The introduction of the commutant with respect to the fermion representation provides us with a natural parametrization of the Dirac–Lagrangian which is compatible with the geometrical scheme of a GTDT. Indeed, the Dirac potential is but a trace of an endomorphism which is uniquely determined by  $D$  (cf section 2.2 in [TT05] and Prop. 3.1 in [Tol07]). As a result, the Dirac potential (2) is replaced by

$$V_{D,\mathfrak{Z}} \equiv \text{tr}(\mathfrak{Z} r_M) + \text{tr}(\mathfrak{Z} \gamma^{\mu\nu} [H_\mu, H_\nu]) + \frac{1}{8} g_{\mu\nu} \text{tr}(\mathfrak{Z} \gamma^\sigma [H_\sigma, \gamma^\mu] \gamma^\lambda [H_\lambda, \gamma^\nu]). \tag{49}$$

Of course, in the more simple case where  $\rho_F$  is irreducible, this replacement just leads back to  $V_{D,\mathfrak{Z}} = \lambda_D V_D$ .

The parametrization of  $D$  depends on its specific form. In general, one may replace  $A_\mu$  by  $A'_\mu \equiv \lambda'_A A_\mu$  and (9) by

$$\lambda \Phi_D \equiv \sum_{k=0}^4 \sum_{0 \leq \nu_1 < \dots < \nu_k \leq 3} \gamma^{\nu_1} \dots \gamma^{\nu_k} \lambda_{(k)} \chi_{\nu_1 \dots \nu_k}^{(k)}, \tag{50}$$

with  $\lambda_{(k)}$  being appropriate ‘coupling constants’.

For example, in the case of the PDY this corresponds to the replacement:

$$\theta_\mu \mapsto \theta'_\mu \equiv \lambda' \theta_\mu := \lambda'_A A_\mu + \lambda'_H H_\mu, \tag{51}$$

$$F_{\mu\nu}^\theta \mapsto \lambda^{-1} F_{\mu\nu}^{\theta'} \equiv \lambda_A^{-1} F_{\mu\nu}^{A'} + \lambda_H^{-1} F_{\mu\nu}^{A',H'}. \tag{52}$$

Here, the curvature  $F_{\mu\nu}^{\theta'}$  is defined with respect to the parameterized gauge potentials  $A'_\mu$  and  $H'_\mu \equiv \lambda'_H H_\mu$ . Of course, by re-scaling the gauge potentials one may assume without loss of generality that  $\lambda'_A = \lambda'_H \equiv 1$ . Therefore, the parameterized PDY corresponds to the replacement:

$$F_{\mu\nu}^\theta \mapsto \lambda^{-1} F_{\mu\nu}^\theta \equiv \lambda_A^{-1} F_{\mu\nu}^A + \lambda_H^{-1} F_{\mu\nu}^{A,H}. \tag{53}$$

From a geometrical perspective such a parametrization is quite acceptable, for curvatures are always considered as elements of vector spaces in contrast to the corresponding gauge potentials. Moreover, the re-parametrization (53) is known from the usual geometrical description of Yang–Mills gauge theories. However, the constants  $\lambda_A, \lambda_H$  should not be identified with the usual Yang–Mills-coupling constants. For example, in the case of a simple gauge group  $G$  and an irreducible fermionic representation  $\rho_F$  thereof, the constant  $\lambda_A$  turns out only to be proportional to the Yang–Mills-coupling constant  $g_{YM}$  which parameterizes the most general Killing form of  $G$ . More precisely, one obtains

$$\lambda_A = \sqrt{\frac{\lambda_{YM}}{\lambda_{rep}}} g_{YM} \tag{54}$$

with  $\lambda_{YM}$  being an arbitrary free parameter.

Moreover, as it turns out, one may put  $\lambda_H$  equal to 1 without loss of generality (cf (58) and (59)). However, the constant  $\lambda_A$  will be crucial for the calculation of the Higgs mass (cf relations (76) below). Indeed, this freedom will guarantee the numerical consistence of the presented geometrical description of the STM.

Note that the replacement  $D \mapsto \lambda D$  is mathematically inappropriate for  $D$  belongs to an affine space. Moreover, since the relative curvature  $F_{\mu\nu}^{A,H}$  decomposes into  $F_{\mu\nu}^{A,H} = F_{\mu\nu}^H + \kappa_{\mu\nu}^{A,H}$ , one may introduce the more general parametrization

$$\lambda_H^{-1} F_{\mu\nu}^{A,H} \mapsto \lambda_H^{-1} F_{\mu\nu}^H + \lambda_{\text{int}}^{-1} \kappa_{\mu\nu}^{A,H}. \quad (55)$$

However, for reasons of covariance one has to identify  $\lambda_{\text{int}}$  with  $\lambda_H$ . Finally, the parametrization of the off-diagonal elements of (21) by the same coupling constants(s) is enforced by quantum field theory. In fact, it is well known that the occurrence of the Pauli term in the fermionic part of the Lagrangian spoils the renormalizability of the fermionic theory. It therefore has to drop out in the fermionic action (cf section five in [TT05]).

In the *parameterized form* the Dirac–Lagrangian with respect to the above data of the (minimal) STM explicitly reads

$$\begin{aligned} \mathcal{L}_D &\equiv \mathcal{L}_{D,\text{fer}} + \mathcal{L}_{D,\text{bos}}, \\ \mathcal{L}_{D,\text{fer}} &:= (\bar{\psi} i\gamma^\mu \nabla_\mu^{\text{Cl}} \psi) \sqrt{-|g|} d^4x + i(\bar{\psi} \gamma_5 \phi \psi) \sqrt{-|g|} d^4x, \end{aligned} \quad (56)$$

$$\begin{aligned} \mathcal{L}_{D,\text{bos}} &:= \lambda_{\text{gr}} r_M \sqrt{-|g|} d^4x - \lambda'_{\text{YM}} \text{tr}(\mathfrak{Z} F_{\mu\nu}^{A\dagger} F_A^{\mu\nu}) \sqrt{-|g|} d^4x \\ &\quad + \lambda'_H \text{tr}(\mathfrak{Z} \nabla_\mu \phi^\dagger \nabla^\mu \phi) \sqrt{-|g|} d^4x \\ &\quad - (\alpha'_H \text{tr} \mathfrak{Z} (\phi^\dagger \phi)^2 - \beta'_H \text{tr}(\mathfrak{Z} \phi^\dagger \phi)) \sqrt{-|g|} d^4x, \end{aligned} \quad (57)$$

$$\alpha'_H = \frac{27}{64} \frac{1}{\pi \text{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_H^2}, \quad \beta'_H = \frac{1}{4} \frac{1}{\pi \text{tr}(\mathfrak{Z})} \frac{1}{\ell_P^2}, \quad (58)$$

$$\lambda'_{\text{YM}} = \frac{1}{8} \frac{1}{\pi \text{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_A^2}, \quad \lambda'_H = \frac{9}{32} \frac{1}{\pi \text{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_H^2}. \quad (59)$$

Here,  $\psi$  is already appropriately re-scaled and  $\mathcal{L}_{D,\text{bos}}$  is normalized such that  $\lambda_{\text{gr}} = \frac{1}{16\pi\ell_P^2}$ . Note that there are two independent length-scales involved which are introduced for quite different reasons. First, one arbitrary length-scale  $\ell$  is introduced to provide the various fields involved with the appropriate physical (length) dimension. As a consequence, one has then to introduce a second length-scale that is given by the Planck length  $\ell_P$  (Newton’s gravitational constant), to make the Einstein–Hilbert Lagrangian dimensionless. Note that to identify this second length-scale with the Planck scale is mainly motivated by the ‘Newtonian limit’ of Einstein’s theory of gravity. Otherwise, this second length-scale is also considered as a free parameter. As it turns out, the length-scale  $\ell$  corresponds to the (inverse of the) Higgs mass (please, see below), contrary to what one might naively expect from (58). We stress again that all fields involved are considered to be represented with respect to the fermion representation  $\rho_F$ .

#### 4. Higgs mass relations

Since the STM-Lagrangian has a natural ‘square root’ in terms of a PDY, one obtains specific relations between the corresponding parameters. In the following two sub-sections it will be demonstrated how these relations yield restrictions to the Higgs mass when (one-loop) quantum corrections are taken into account.

4.1. Parameter relations between GTDT and the STM

In this sub-section we first re-write the parameterized Dirac–Lagrangian (56) in terms of the ordinary fields of the STM. A comparison with the usual parametrization of the STM-Lagrangian yields some constraints of the parameters which are not known in the usual description of the minimal Standard Model. This discussion is analogous to what has been presented already in [Tol98]. Hence, we will skip the details and present here only the relevant results. With the help of these results, however, we will show how the geometrical description of the STM in terms of GTDT gives rise to bounds of the Higgs mass. This, of course, will be discussed in some detail in the following sub-section.

The bosonic Lagrangian within the usual description of the STM reads

$$\begin{aligned} \mathcal{L}_{\text{SM}} = & \left( \frac{1}{2g_3^2} \text{tr}(C_{\mu\nu}C^{\mu\nu}) + \frac{1}{2g_2^2} \text{tr}(W_{\mu\nu}W^{\mu\nu}) + \frac{1}{4g_1^2} B_{\mu\nu}B^{\mu\nu} \right) \sqrt{-|g|} d^4x \\ & + \frac{1}{2} (\nabla_\mu \varphi)^* (\nabla^\mu \varphi) \sqrt{-|g|} d^4x - \left[ \lambda (\varphi^* \varphi)^2 - \frac{\mu^2}{2} \varphi^* \varphi \right] \sqrt{-|g|} d^4x. \end{aligned} \tag{60}$$

Here, respectively,  $C_{\mu\nu}$ ,  $W_{\mu\nu}$  and  $B_{\mu\nu}$  denote the Yang–Mills field strengths with respect to the fundamental representation of  $SU(3)$ ,  $SU(2)$  and  $U(1)$ ,  $\varphi$  is the usual Higgs doublet sitting in the fundamental representation of  $SU(2)$ . Its  $U(1)$  representation is defined with respect to the hypercharge relations (37), which turn out to be crucial for re-writing (57) in terms of the physical fields.

In terms of the data of the (minimal) Standard Model the Dirac–Lagrangian of a PDY (57) corresponds to the bosonic STM-Lagrangian (60) provided the following relations are fulfilled (see also [Tol98, Thu03]):

$$\frac{1}{g_1^2} = 2A \frac{3Ny_q + y_l \text{tr} X}{4N + \text{tr} X}, \quad \frac{1}{g_2^2} = A \frac{3N + \text{tr} X}{4N + \text{tr} X}, \quad \frac{1}{g_3^2} = 4A \frac{N}{4N + \text{tr} X}, \tag{61}$$

$$1 = 2B \frac{3 \text{tr}(\mathbf{g}^{q'} \mathbf{g}^{q'*} + \mathbf{g}^q \mathbf{g}^{q*}) + \text{tr}(X \mathbf{g}^l \mathbf{g}^{l*})}{4N + \text{tr} X}, \tag{62}$$

$$\lambda = C \frac{3 \text{tr}((\mathbf{g}^{q'} \mathbf{g}^{q'*})^2 + (\mathbf{g}^q \mathbf{g}^{q*})^2) + \text{tr}(X (\mathbf{g}^l \mathbf{g}^{l*})^2)}{4N + \text{tr} X}, \tag{63}$$

$$\mu^2 = \frac{1}{3\pi} \left( \frac{1}{\ell_P} \right)^2 \frac{3 \text{tr}(\mathbf{g}^{q'} \mathbf{g}^{q'*} + \mathbf{g}^q \mathbf{g}^{q*}) + \text{tr}(X \mathbf{g}^l \mathbf{g}^{l*})}{4N + \text{tr} X}. \tag{64}$$

Here, we used the following abbreviations:

$$y_q := 2(y_L^q)^2 + (y_R^d)^2 + (y_R^u)^2, \quad y_l := 2(y_L^l)^2 + (y_R^l)^2, \tag{65}$$

$$X := \frac{\lambda^l}{\lambda_q}, \quad A := \frac{1}{12\pi} \left( \frac{\ell}{\ell_P} \right)^2 a^2, \tag{66}$$

$$B := \frac{1}{3\pi} \left( \frac{\ell}{\ell_P} \right)^2 b^2, \quad C := \frac{1}{2\pi} \left( \frac{\ell}{\ell_P} \right)^2 b^2,$$

with  $a := \frac{1}{\lambda_\Lambda}$ ,  $b := \frac{3}{4} \frac{1}{\lambda_H}$  and  $\lambda_q \in \mathbb{R}^+$ , as well as  $\lambda^l := \text{diag}(\lambda_{l1}, \dots, \lambda_{lN})$  and  $\lambda_{il} \in \mathbb{R}^+$ . We also made use of  $\mathfrak{J} = \mathbf{1}_4 \otimes \text{diag}(z_L, z_R)$ , with  $z_L := \text{diag}(\lambda_q \mathbf{1}_{6N}, \lambda^l \otimes \mathbf{1}_2)$  and  $z_R := \text{diag}(\lambda_q \mathbf{1}_{6N}, \lambda^l)$  and of relations (37).

From relations (62) and (64) one immediately infers that

$$\ell_H \equiv \frac{\ell}{\lambda_H} = 2\sqrt{\frac{2}{3}} \frac{1}{\mu}. \tag{67}$$

**Table 1.** Gauge couplings

	Value	Abs. error
$g_1$	0.345 37	0.000 03
$g_2$	0.629 76	0.000 20
$g_3$	1.221 32	0.002 90

Values of the gauge couplings at the energy  $E_0 = m_Z = 91.1876 \pm 0.0021$  GeV, cf [Y06a].

As mentioned already, we may put  $\lambda_H \equiv 1$  for it simply re-scales  $\ell \mapsto \ell_H$  according to relations (58) and (59). As a consequence, a PDY is physically parameterized by the two constants  $(\ell_H, \lambda_{YMH})$  instead of the three parameters  $(\ell, \lambda_A, \lambda_H)$  with  $\lambda_{YMH} \equiv \lambda_H/\lambda_A$ . Moreover, as far as the Standard Model is concerned, the ‘relative-coupling constant’  $\lambda_{YMH}$  turns out to be numerically fixed (see below).

Because of (67), one has

$$\ell_H \sim 1/m_H. \quad (68)$$

Hence, the two length-scales involved in the geometrical description of the (minimal) Standard Model as a specific GTDT are determined by the Planck mass and the mass of the Higgs. As one may naively expect, it turns out that  $m_H/m_P \lll 1$  on one-loop and top-quark-mass approximation. Therefore, within these approximations  $\ell_H$  is the dominant length-scale and gravitational effects may be fully negligible.

Relations (61)–(64) are derived on ‘tree-level’ by comparing (57) with (60). In this approximation, however, the constraints for the gauge couplings (61) are inconsistent with the known experimental data (cf table 1). More precisely, when taking into account the measured values of the gauge couplings there exists no choice of the model parameters such that all three relations for the gauge couplings are fulfilled. On the other hand, it is well known that the gauge couplings are running couplings which depend on the considered energy scale. Hence, according to the renormalization group philosophy, the inconsistency of (61) may be interpreted in such a way that (61)–(64) are actually supposed to hold true only at certain critical values of the energy scale. In the following sub-section we will make use of the renormalization flow equations to determine these critical energy values. At the critical values it is then possible to solve the parameter relations with respect to the Higgs self-coupling  $\lambda$  from which we eventually obtain the Higgs mass via the ratio

$$\frac{m_H}{m_W} = \frac{\sqrt{16\lambda}}{g_2}, \quad (69)$$

where the numerical values of the gauge coupling  $g_2$  and the mass of the W-boson are regarded to be known from experiments.

#### 4.2. One-loop quantum corrections and the Higgs mass

In this section we follow the same strategy as in [CIKS97, CIS97] to determine the mass of the Higgs boson.

For the STM the renormalization flow equations in one-loop and top-quark-mass approximation have been derived in [FJSE93] using the  $\overline{MS}$  scheme:

$$\dot{g}_1 = \beta_1(g_1) := \frac{41}{96\pi^2} g_1^3, \quad \dot{g}_2 = \beta_2(g_2) := -\frac{19}{96\pi^2} g_2^3, \quad \dot{g}_3 = \beta_3(g_3) := -\frac{7}{16\pi^2} g_3^3, \quad (70)$$

$$\dot{g}_t = \beta_t(g_1, g_2, g_3, g_t) := \frac{1}{16\pi^2} \left( 9g_t^3 - \left( 8g_3^2 + \frac{9}{4}g_2^2 + \frac{17}{12}g_1^2 \right) g_t \right), \quad (71)$$

$$\begin{aligned} \dot{\lambda} &= \beta_\lambda(g_1, g_2, g_3, g_t, \lambda) \\ &:= \frac{1}{16\pi^2} \left( 96\lambda^2 + (24g_t^2 - 9g_2^2 - 3g_1^2)\lambda - 6g_t^4 + \frac{9}{32}g_2^4 + \frac{3}{32}g_1^4 + \frac{3}{16}g_1^2g_2^2 \right). \end{aligned} \quad (72)$$

Here, the derivative is taken with respect to a dimensionless scale parameter  $t = \ln\left(\frac{\Lambda}{E_0}\right)$ , with  $\Lambda$  being an arbitrary energy scale and  $E_0$  reference energy.  $g_t$  is the Yukawa coupling of the top quark.

The renormalization flow equations for  $g_1, g_2, g_3$  can be explicitly integrated:

$$g_1(t) = \frac{1}{\sqrt{A_1 - \frac{41}{48\pi^2}t}}, \quad g_2(t) = \frac{1}{\sqrt{A_2 + \frac{19}{48\pi^2}t}}, \quad g_3(t) = \frac{1}{\sqrt{A_3 + \frac{7}{8\pi^2}t}}, \quad (73)$$

where  $A_i \equiv 1/g_i^2(0)$ ,  $i = 1, \dots, 3$ .

We may use these solutions to determine the critical values  $t_c$  for which relations (61)–(64) are fulfilled. When relations (38) of the hyper-charges are taken into account one actually obtains a unique value of the critical energy ( $N = 3$  generations)

$$t_c = \frac{8\pi^2}{21}(3A_1 - 9A_2 + 4A_3) \quad (74)$$

$$= 19.322\,539\,88 \pm 0.121\,725\,5988, \quad (75)$$

which corresponds to  $E_c = (2.247 \pm 0.274) \times 10^{10}$  GeV. Here, we have again made use of the experimental values summarized in table 1.

Next, we aim to find an initial value for  $\lambda$  at the critical scale point  $t_c$  and integrate system (71) and (72). This allows us to compute the value of  $\lambda$  at any scale point  $t$  where the values for  $g_2$  and  $m_W$  are known.

If one divides (63) and the third equation of (61) by (62) one obtains in the top-quark-mass approximation

$$\lambda = \frac{3}{4}g_t^2, \quad \lambda_{\text{YMH}}^2 = \frac{9}{8}\frac{g_t^2}{g_3^2}. \quad (76)$$

Note that the parameter  $\lambda_{\text{YMH}}$  is numerically fixed by the  $SU(3)$  Yang–Mills-coupling constant  $g_3$  and the Yukawa-coupling constant  $g_t$  of the top quark at the critical scale point  $t_c$  (resp. at the critical energy  $E_c$ ).

To proceed, we numerically integrate the system of differential equations (71) and (72). First, we integrate the differential equation (71) for  $g_t$  with respect to the initial value at  $t = 0$ . For this, let  $t = 0$  correspond to the reference energy  $E_0 = m_Z = 91.1876 \pm 0.0021$  GeV. We then calculate the initial values for the top-Yukawa coupling  $g_t$  from the top mass with help of the relation

$$\frac{g_t}{g_2} = \frac{1}{2}\frac{m_T}{m_W} \quad (77)$$

(cf [CIKS97]). With the data taken from table (1) and  $m_W = 80.403 \pm 0.029$  GeV from [Y063c] and the quark masses from [Y062b]:

$$m_T = 174.2 \pm 3.3 \text{ GeV} \quad (\text{direct observation of top events}), \quad (78)$$

$$m_T = 172.3 \pm 10.2 \text{ GeV} \quad (\text{Standard Model electroweak fit}) \quad (79)$$

we obtain as initial values

$$g_t(0) = 0.682\,214\,2710 \pm 0.013\,391\,043\,34 \quad (\text{direct observation of top events}), \quad (80)$$

$$g_t(0) = 0.674\,773\,3575 \pm 0.040\,408\,210\,40 \quad (\text{Standard Model electroweak fit}). \quad (81)$$

With respect to this solution the value of  $g_t$  can be computed at the scale point  $t_c$ . Next, we numerically integrate (72) with respect to the initial condition

$$\lambda(t_c) = \frac{3}{4}g_t(t_c). \quad (82)$$

Finally, this allows us to calculate  $\lambda = \lambda(0)$  for the reference energy  $E_0 = m_Z = 91.187$  GeV (cf [Y062b]):

$$\lambda(0) = 0.066\,1110 \pm 0.005\,4824 \quad (\text{direct observation of top events}), \quad (83)$$

$$\lambda(0) = 0.064\,7427 \pm 0.015\,9511 \quad (\text{Standard Model electroweak fit}). \quad (84)$$

As a consequence, we obtain for the Higgs mass the values

$$m_H = 185.6990 \pm 7.6789 \text{ GeV} \quad (\text{direct observation of top events}), \quad (85)$$

$$m_H = 183.7671 \pm 21.4054 \text{ GeV} \quad (\text{Standard Model electroweak fit}) \quad (86)$$

for  $m_W = 80.403 \pm 0.029$  GeV (cf [Y062b, Y063c]). Here, the error is due to the errors of the initial values of  $g_1, g_2, g_3, t_c$  and  $m_W, m_T$  as well as their influence on the numerical integration of the renormalization flow equations.

According to the STM the Higgs mass can be restricted to the interval (cf [Ros03]):

$$m_H \in [114, 193] \text{ GeV}. \quad (87)$$

It is valid for the STM with one Higgs boson and without super-symmetry. As a result, the predicted value of the Higgs mass within the geometrical frame of GTDT is found to be at the upper bound of the interval (87).

#### 4.3. Model bounds for the Higgs mass and massive neutrinos

For the sake of completeness (and comparison, see below), we briefly discuss here how the statement (85) may be weakened if one introduces the most general parametrization of the PDY. We stress, however, that such a parametrization is not favored by the geometrical setup of GTDT (cf [TT05]). Disregarding geometry, however, such a non-geometrical parametrization may be still of interest for it yields the principal model bounds for the predicted value of the Higgs mass within the mathematical frame presented. In this sub-section, we also briefly discuss how the statement (85) may depend on massive neutrinos.

From the naive point of view of ‘counting free parameters’ one may parameterize (53) also as follows:

$$\begin{aligned} \lambda^{-1} F_{\mu\nu}^\theta &\equiv \lambda_A^{-1} F_{\mu\nu}^A + \lambda_H^{-1} (\partial_\mu H_\nu - \partial_\nu H_\mu + [A_\mu, H_\nu] - [A_\nu, H_\mu]) + \lambda_{\text{self}}^{-1} [H_\mu, H_\nu] \\ &= \lambda_A^{-1} F_{\mu\nu}^A + \lambda_H^{-1} ([\nabla_\mu^A, H_\nu] - [\nabla_\nu^A, H_\mu]) + \lambda_{\text{self}}^{-1} [H_\mu, H_\nu]. \end{aligned} \quad (88)$$

It turns out that this non-geometrical parametrization of the PDY does not change the critical scale point  $t_c$ . In particular, (88) does not alter the uniqueness of the critical energy point  $E_c$ . Nonetheless, the parametrization (88) gives rise to a ‘fuzziness’ of the predicted Higgs mass, similar to what is obtained within the ‘(real) Connes–Lott’ description of the STM (please, see the following section).

To obtain the general bounds for the Higgs mass we have to consider the Dirac–Lagrangian for PDY’s with the Pauli term being parameterized like (88). One gets the same expressions



as in (56) and (57), however, with different pre-factors. In the case considered they read

$$\alpha'_H = \frac{27}{64} \frac{1}{\pi \operatorname{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_{\text{self}}^2}, \quad \beta'_H = \frac{1}{4} \frac{1}{\pi \operatorname{tr}(\mathfrak{Z})} \frac{1}{\ell_P^2}, \quad (89)$$

$$\lambda'_{\text{YM}} = \frac{1}{8} \frac{1}{\pi \operatorname{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_A^2}, \quad \lambda'_H = \frac{9}{32} \frac{1}{\pi \operatorname{tr}(\mathfrak{Z})} \left(\frac{\ell}{\ell_P}\right)^2 \frac{1}{\lambda_H^2}. \quad (90)$$

These relations differ from (58) and (59) only by the self-coupling constant  $\alpha'_H$  due to the parametrization (88).

By the same analysis as for the geometrically parameterized PDY one obtains parameter relations analogous to (61)–(64). They only differ from the latter in the definition of the constants  $A, B, C$  :

$$A := \frac{1}{12\pi} \left(\frac{\ell_H}{\ell_P}\right)^2 \lambda_{\text{YMH}}^2, \quad B := \frac{3}{16\pi} \left(\frac{\ell_H}{\ell_P}\right)^2, \quad C := \frac{9}{32\pi} \left(\frac{\ell_H}{\ell_P}\right)^2 \lambda_{\text{H,self}}^2 \quad (91)$$

where again  $\ell_H \equiv \ell/\lambda_H$ ,  $\lambda_{\text{YMH}} \equiv \lambda_H/\lambda_A$  and  $\lambda_{\text{H,self}} := \lambda_H/\lambda_{\text{self}}$ .

Doing the same analysis as in the previous section one concludes that the value for the critical scale point does not change. Basically, the reason is that the relations for the gauge couplings (61) remain the same. For this reason one may proceed in the same way as before to end up with

$$\lambda_{\text{YMH}}^2 = \frac{9}{8} \frac{g_t^2}{g_3^2}, \quad \lambda = \frac{3}{4} \lambda_{\text{H,self}}^2 g_t^2. \quad (92)$$

Therefore, with respect to the more general parametrization  $(\ell_H, \lambda_{\text{YMH}}, \lambda_{\text{H,self}})$  the value of the Higgs self-coupling constant  $\lambda$  at the critical scale point is not fixed by the appropriate value of  $g_t(t_c)$ . In other words, unlike to the geometrical parametrization, essentially defined by the length scale  $\ell_H \simeq 1/m_H$ , the non-geometrical parametrization gives rise to an additional coupling constant  $\lambda_{\text{H,self}}$ , that is also tied to the Higgs mass.

In what follows we abbreviate  $\kappa := \frac{3}{4} \lambda_{\text{H,self}}^2 g_t^2$ . We now have to integrate the flow equation (72) with respect to the initial value

$$\lambda(t_c) = \kappa, \quad \kappa > 0. \quad (93)$$

The model bounds are then determined by the boundary values for (71) with respect to (93), which give rise to the minimal and the maximal values for the Higgs mass.

Due to standard theorems on ordinary differential equations the values of the solutions at a certain scale point  $t$  of (72) depend monotonically on the initial value (cf [Ama83]). Hence, the lower bound for the Higgs mass is determined by integrating (71) and calculating  $\underline{m}_H$  with respect to (69) at  $\lambda(t_c) = 0$ . One gets for the considered cases for the top mass

$$\underline{m}_H = 130.952 \text{ GeV} \quad (\text{direct observation of top events}), \quad (94)$$

$$\underline{m}_H = 113.142 \text{ GeV} \quad (\text{Standard Model electroweak fit}). \quad (95)$$

In order to obtain an upper bound for the Higgs mass one looks for a differential equation

$$\dot{\lambda} = \tilde{\beta}_\lambda \quad (96)$$

such that

- (1) for the same initial values  $\kappa$  the solutions of this equation are upper bounds of the solutions of (72);
- (2) the solutions explicitly depend on the initial value  $\kappa$ .

The second property permits us to calculate the limit  $\kappa \rightarrow +\infty$  which yields the upper bound for all possible solutions of (72).

One may define  $\tilde{\beta}_\lambda$  in the following way:

$$\begin{aligned}\tilde{\beta}_\lambda &:= \ell_1 \lambda^2 + \ell_2 \lambda + \ell_3, \\ \ell_1 &:= \frac{6}{\pi^2}, \quad \ell_2 := \frac{1}{16\pi^2} (24\overline{g}_t^2(0) - 9\overline{g}_2^2(t_c + \Delta t_c) - 3\overline{g}_1^2(0)), \\ \ell_3 &:= \frac{1}{16\pi^2} \left( -6\overline{g}_t^4(t_c + \Delta t_c) + \frac{9}{32}\overline{g}_2^4(0) + \frac{3}{32}\overline{g}_1^4(t_c + \Delta t_c) + \frac{3}{16}\overline{g}_2^2(0)\overline{g}_1^2(t_c + \Delta t_c) \right).\end{aligned}\quad (97)$$

Here,  $\Delta t_c$  is the error of the critical scale point  $t_c$ . We also used the following abbreviations:

$$\begin{aligned}\overline{g}_1(t) &:= \frac{1}{\sqrt{A_1 - \Delta A_1 - \frac{41}{48\pi^2}t}}, & \overline{g}_2(t) &:= \frac{1}{\sqrt{A_2 - \Delta A_2 + \frac{18}{48\pi^2}t}}, \\ \overline{g}_3(t) &:= \frac{1}{\sqrt{A_3 - \Delta A_3 + \frac{8}{7\pi^2}t}},\end{aligned}\quad (98)$$

and

$$\begin{aligned}\underline{g}_1(t) &:= \frac{1}{\sqrt{A_1 + \Delta A_1 - \frac{41}{48\pi^2}t}}, & \underline{g}_2(t) &:= \frac{1}{\sqrt{A_2 + \Delta A_2 + \frac{18}{48\pi^2}t}}, \\ \underline{g}_3(t) &:= \frac{1}{\sqrt{A_3 + \Delta A_3 + \frac{8}{7\pi^2}t}}, \\ \Delta A_i &:= 2 \frac{\Delta g_i}{g_{i,0}^3}, \quad i = 1, \dots, 3,\end{aligned}\quad (99)$$

with  $\Delta g_i$  being the error of the initial value  $g_{i,0}$  (cf table 1).

Here,  $\overline{g}_t(t)$  is the solution of the initial value problem

$$\dot{g}_t(t) = \frac{1}{16\pi^2} \left( 9g_t^3 - \left( 8\overline{g}_3^2 + \frac{9}{4}\overline{g}_2^2 + \frac{17}{12}\overline{g}_1^2 \right) g_t \right), \quad g_t = g_{t0} + \Delta g_t, \quad (100)$$

and  $\underline{g}_t$  is the solution of the initial value problem

$$\dot{g}_t(t) = \frac{1}{16\pi^2} \left( 9g_t^3 - \left( 8\underline{g}_3^2 + \frac{9}{4}\underline{g}_2^2 + \frac{17}{12}\underline{g}_1^2 \right) g_t \right), \quad g_t = g_{t0} - \Delta g_t, \quad (101)$$

with  $\Delta g_t$  being the error of the initial value  $g_{t0}$  (cf (80) and (81)).

Any solution  $\tilde{\lambda}$  of (96) with initial value  $\kappa > 0$  fulfils  $\tilde{\lambda}(t) \geq \lambda(t)$  for  $t \geq 0$ , with  $\lambda(t)$  being the solution of (72) with initial value  $\kappa$ . Using (97), the differential equation (96) can be explicitly solved for arbitrary initial value  $\kappa$ :

$$\begin{aligned}\tilde{\lambda}(t) &= \sqrt{|\beta|} \coth \left( \frac{\ell_1 \beta}{\sqrt{|\beta|}} (t - t_c) + \operatorname{arccoth} \left( \frac{\kappa + \alpha}{\sqrt{|\beta|}} \right) \right) - \alpha, \\ \alpha &:= \frac{1}{2} \frac{\ell_1}{\ell_2}, \quad \beta := -\frac{1}{4} \frac{\ell_2^2}{\ell_1^2} + \frac{\ell_3}{\ell_1}, \quad \beta < 0.\end{aligned}\quad (102)$$

This permits us to calculate  $\tilde{\lambda}_{as}(t) := \lim_{\kappa \rightarrow \infty} \tilde{\lambda}(t)$ :

$$\tilde{\lambda}_{as}(t) := \sqrt{|\beta|} \coth \left( \frac{\ell_1 \beta}{\sqrt{|\beta|}} (t - t_c) \right) - \alpha, \quad (103)$$

which finally yields the upper bound  $\overline{m_H}$  for the Higgs mass at the scale point  $t = 0$  :

$$\tilde{\lambda}_{as}(0) = 0.463\,874\,798 \quad \Rightarrow \quad \overline{m_H} = 492.232\,373 \text{ GeV} \quad (\text{direct observation top events}), \quad (104)$$

$$\tilde{\lambda}_{as}(0) = 0.459\,830\,097 \quad \Rightarrow \quad \overline{m_H} = 490.081\,276 \text{ GeV} \quad (\text{SM electroweak fit}). \quad (105)$$

Therefore, we end up with the following range of the predicted value of the Higgs mass:

$$m_H \in [130.95, 492.23] \text{ GeV} \quad (\text{direct observation of top events}), \quad (106)$$

$$m_H \in [113.14, 490.08] \text{ GeV} \quad (\text{Standard Model electroweak fit}). \quad (107)$$

Obviously, this range has a non-empty intersection with (87). In particular, it has a lower bound close to the expected value of the Higgs mass. Note that the upper bound corresponds to a rough estimate, only.

Next, we discuss a simple modification of the STM which takes into account the possibility of *massive neutrinos*. We restrict ourselves to a few remarks concerning so-called ‘Dirac-type mass terms’ which seems to fit best with GTDT.

The STM can easily be enhanced with a right-handed neutrino sector by replacing  $\rho_{F,R}$  in (30) as follows:

$$\rho(c, w, \theta) := \text{diag}(c \otimes \mathbf{1}_N \otimes \text{diag}(e^{iy_R^d \theta}, \mathbf{1}_N \otimes \text{diag}(e^{iy_R^e \theta}, e^{iy_R^v \theta})), \quad (108)$$

with, respectively,  $y_R^e, y_R^v \in \mathbb{Q}$  being the hyper-charges of the right-handed electron and neutrino. Accordingly, the matrix  $\tilde{\phi}$  in (35) has to be modified by

$$\tilde{\phi} := \text{diag} \left( \mathbf{1}_3 \otimes \begin{pmatrix} \mathbf{g}^q \varphi_1 & \mathbf{g}^q \bar{\varphi}_2 \\ \mathbf{g}^q \varphi_2 & -\mathbf{g}^q \bar{\varphi}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{g}^l \varphi_1 & \mathbf{g}^l \bar{\varphi}_2 \\ \mathbf{g}^l \varphi_2 & -\mathbf{g}^l \bar{\varphi}_1 \end{pmatrix} \right), \quad (109)$$

where  $\mathbf{g}^l$  and  $\mathbf{g}^q$  are the corresponding leptonic Yukawa-coupling matrices.

This simple modification of the STM permits us to construct a theory that also contains massive neutrinos. The appropriate neutrino mass terms are generated by so-called *Dirac-type mass terms*. It is known, however, that there exist other neutrino mass generating mechanisms as well (cf [Bil02, BGGM03]).

Since Dirac-type mass terms result by only modifying the (right-handed) representation of the gauge group of the STM, one may perform exactly the same analysis as has been carried out in the foregoing section. It turns out that the critical energy scale  $t_c$  (and hence  $E_c$ ) is identical with (75). Moreover, since relations (76) are unchanged one ends up with the same predicted value of the Higgs mass (85) and (86) as in the STM without massive neutrinos. Note that this is indeed remarkable, for the parameter relations which correspond to (61)–(64) are nonetheless different from the relations obtained in the case of massless neutrinos.

## 5. A brief comparison with NCG

As mentioned already in the introduction there are various different geometrical descriptions of the STM. Some of these were especially addressed to make predictions of the Higgs mass. Therefore, it may be also of interest to briefly discuss how some of these approaches to the STM are related to the frame presented here. In what follows, we will restrict ourselves to two different geometrical descriptions of the STM within the general frame of non-commutative geometry. One of which is usually referred to as ‘Chamseddine–Connes model (CCM)’ (see, for example [CC97, CIKS97, CIS97] and for a version including massive neutrinos

[CCM06]) and which has some formal similarity to GTDT. The second approach (which can be actually regarded as the predecessor of CCM) is called the ‘Connes–Lott model (CLM)’ (see, for example [CL90, SZ95, KS97, IKS95a, IKS95b, CIS99, CIS97, IS96]). This model is based on Connes’ general ideas of non-commutative geometry as presented, for example, in [Con94, GBVF01, SZ95].

In the following we mainly discuss what is referred to as the ‘soft version’ of either of these approaches to the STM. The soft version of CLM and CCM (in contrast to the so-called stiff version) also takes into account the possibility of a non-trivial parametrization. Hence, it is more appropriate to compare these versions of CLM and CCM with the presented frame of GTDT. Moreover, the non-parameterized (stiff) versions seem physically inappropriate analogous to ordinary Yang–Mills theory when the gauge-coupling constant is chosen to be equal to 1. Such a non-parameterized version is admissible only with respect to a purely mathematical discussion of the corresponding geometrical scheme. Indeed, a non-parametrization usually yields contradictions with experiments, for it physically corresponds to set (at least some of) the admissible free parameters of a physical theory equal to 1 (see, for instance, [CIS99]). Of course, a specific geometrical setup gives sever restrictions to the admissible parametrization of the scheme considered. Hence, different geometrical descriptions of the same physical theory (like the STM) may also yield different constraints on the corresponding parameter set.

In this section, the cited values for the Higgs mass almost exclusively refer to the older assumed value of the top mass of about 175 GeV. In order to compare the different NCG approaches with the geometrical approach proposed in this paper, we mention that in the case of  $m_T = 175 \pm 6$  GeV the predicted value of the Higgs mass within GTDT reads

$$m_H = 188 \pm 15 \text{ GeV}. \quad (110)$$

Note that this value also refers to the older values of, respectively, the W-mass  $m_W = 80.33 \pm 0.15$  GeV and  $g_1 = 0.3575$ ,  $g_2 = 0.6507$ ,  $g_3 = 1.218$  at  $E_0 = m_Z = 91.187$  GeV. The appropriate critical energy scale is given by  $E_c = 0.96 \times 10^{10}$  GeV.

### 5.1. Comparison with the CCM

The formal similarity in the geometrical description of the STM between CCM and GTDT is that in both approaches Dirac–Yukawa-type operators play a basic role. The motivation, however, is very different. Within the frame of CCM these generalized Dirac operators are motivated by non-commutative geometry (via the tensor product of spectral triples). In contrast, in GTDT the Dirac–Yukawa-type Dirac operators naturally arise from the Bochner–Lichnerowicz–Weitzenböck decomposition. Physically, these Dirac operators are motivated by perturbation theory and the Yukawa coupling. The basic difference of both approaches lies in the ‘action’. Indeed, CCM postulates what is referred to as *spectral action* which incorporates gravity within non-commutative geometry (cf [CC97]). Basically, the evaluation of the spectral action consists of a (sophistically) modified heat kernel asymptotic (see, for instance, in [GBVF01]) up to the second non-trivial coefficient including the ‘cosmological constant’ and quadratic Riemannian curvature terms. As a consequence, the STM Lagrangian is only reproduced if the base manifold is assumed to be flat and the cosmological constant is disregarded. Moreover, for the heat expansion to make mathematical sense one has to deal with (closed) compact Riemannian manifolds instead of (open) Lorentzian manifolds to geometrically model ‘spacetime’ (see also [Sakh75, Sakh82]). In contrast, GTDT only uses (globally defined) densities instead of functionals. Moreover, in the latter scheme a specific Lagrangian (1) is canonically associated with every Dirac operator (which, in particular, may

have arbitrary signature, cf [TT05]). This Lagrangian is fully determined by the Dirac operator in question.

The evaluation of the (soft) spectral action with respect to the Dirac–Yukawa operator that is defined by (29)–(37) leads to parameter relations which are similar to (61)–(64), see, for example, [CIKS97]). In this reference, also the value of the Higgs mass is calculated by a similar analysis to that presented in the previous section. It turns out that the parameter relations for the gauge couplings (61) are equivalent to those derived within CCM. The reason for this is that these relations basically follow from the fermion representation of the gauge fields. This, however, is supposed to hold true in both descriptions of the STM. As a consequence, one obtains the same critical scale point (75) and hence also the same critical energy  $E_c$ . On the other hand, all other parameter relations turn out to be essentially different from those presented here. As a consequence, one obtains a different value for the Higgs mass (see again [CIKS97], as well as [CIS99])

$$m_H = 190 \pm 5 \text{ GeV} \quad (\text{soft action}), \quad (111)$$

where  $m_T = 175 \pm 6 \text{ GeV}$  and  $E_c = 0.96 \times 10^{10} \text{ GeV}$ .

Unfortunately, these predictions of the Higgs mass are of limited value insofar as the CCM approach to the STM is incompatible with certain experimentally known values of the (ratio of the) gauge- and Yukawa-coupling constants. Indeed, the parameter relations of the CCM (for the so-called stiff action) imply the following relation at the critical scale point  $t_c$  between the  $SU(3)$ -gauge-coupling constant and the Yukawa-coupling constant of the top quark:

$$\frac{g_3^2}{g_t^2} = \frac{3}{2}. \quad (112)$$

This relation, however, is not compatible with the known experimental data. A similar numerical inconsistency is obtained also in the case of the so-called ‘soft-action’ (see again [CIKS97]; a detailed discussion may be found in [Thu03]). The same holds true with respect to the value  $m_H = 175_{-7.8}^{+5.8}$  (with  $m_T = 178.0 \pm 6 \text{ GeV}$  and  $E_c = 1.1 \times 10^{17} \text{ GeV}$ ) for the Higgs mass presented in [KS06], since this value also refers to the CCM (cf also the discussion about the role of gravity in the conclusion).

In [CCM06] the authors discuss the inclusion of Majorana spinors in the CCM approach to the STM. From an analysis similar to the one presented here, the authors arrive at a predicted Higgs mass

$$m_H \approx 168\text{--}170 \text{ GeV}. \quad (113)$$

The concrete value depends on the specific assumptions made on the expected mass of the neutrinos. For example, the value  $m_H \approx 170 \text{ GeV}$  holds if all but the top mass is neglected. The corresponding discussion parallels that already presented in [KS06]. In contrast, if one of the neutrino masses is supposed to be of the order of the top mass, then  $m_H \approx 168 \text{ GeV}$ .

However, value (113) of the Higgs mass only follows from the ‘stiff’ action of the CCM. As a consequence, the calculated parameter relations imply the GUT relations between the Yang–Mills-coupling constants:

$$g_3^2 = g_2^2 = \frac{5}{3} g_1^2. \quad (114)$$

As the authors remark, these relations are known to contradict the measured values of the coupling constants at the W-mass scale. Indeed, the renormalization flow does not yield relations (114) at the GUT energy scale when the measured values of the gauge-coupling constants at the W-mass energy scale are taken into account as initial conditions. This is usually interpreted in such a way that there is no ‘big desert’. However, the existence of

a big desert is a basic assumption in the CCM and CLM approach to the STM (like in the geometrical description presented in this work, cf our remarks in the introduction). Actually, in [CCM06] the authors do not take into account the second equality in (114) in order to calculate the value of the Higgs mass (113). Indeed, the first equality in (114) yields a unique critical energy scale of about  $E_c \sim 10^{17}$  GeV, being seven order of magnitudes higher than the critical energy scale in GTDT. From the corresponding discussion in [KS06], it follows that the higher the critical energy scale (i.e. the bigger the desert), the lower the predicted value of the Higgs mass within NCG.

Besides the numerical inconsistency that is implied by the full GUT relations (114), one may also ask for the ‘inner consistence’, for example, with respect to the assumption that gravitational effects are negligible also on an energy scale that is only two order of magnitudes below the Planck scale (cf also our corresponding discussion at the end of the paper). Nonetheless, one may infer from the results presented in [CCM06] that adding massive neutrinos to the STM seems to (slightly) lower the value of the Higgs mass.

The inclusion of neutrino masses, as presented in [CCM06], may also have the interesting feature to remedy the numerical inconsistency (112) in the CCM approach to the STM. Relation (112) actually depends on the assumptions made about the value of the neutrino masses. Hence, one may ask for the order of magnitude of the neutrino masses that is necessary to overcome the numerical inconsistency caused by relation (112). Let us recall that the latter relation is derived under the assumption of the validity of the top-quark-mass approximation.

In contrast to relation (112), in the geometrical frame presented the above relation between the Yukawa-coupling constant and the  $SU(3)$ -gauge-coupling constant is replaced by (76) which includes the free ‘relative-coupling constant’  $\lambda_{\text{YMH}}$ . This additional free parameter has its origin in the generalization of the usual Yang–Mills curvature. It is quite remarkable that the tree-level relations (58) and (59) are such that only  $\lambda_{\text{H}}$ , but not  $\lambda_{\text{A}}$ , can be chosen equal to 1 without loss of generality. This is because of the usual Yang–Mills term in the bosonic Lagrangian (57). It is this subtle interplay between the usual Yang–Mills curvature and its generalization with respect to the Higgs gauge potential which allows the presented geometrical description of the STM to be numerically consistent.

## 5.2. Comparison with the CLM

The Connes–Lott approach to the Standard Model is clearly conceptually different from the geometrical frame presented here. The CLM essentially incorporates the basic ideas of Connes’ mathematical theory of non-commutative geometry. Hence, it does not come as a surprise that the appropriate parameter relations obtained from the CLM are basically different from those implied by the GTDT approach to the STM (cf [CIS99]). Yet, in both geometrical schemes Dirac-type operators of the form (19) play a fundamental role though their geometrical origin and physical interpretation are quite different. Indeed, in the CLM the geometrical role of the operators (19) is two-fold: first, they correspond to total exterior derivatives (in this context  $\mathcal{D}$  is referred to as the ‘inner Dirac operator’); second, (19) induces the non-commutative analogue of the Riemannian volume measure  $\mu_{\text{M}}$  in the bosonic action (Dixmier trace). However, the fermionic and the bosonic CLM-action are defined in totally different ways, in contrast to the CCM and the frame presented here. In any case, in the CLM the (Riemannian) metric has to be chosen by hand, similar to the case of Yang–Mills gauge theories. Actually, the bosonic action in the CLM frame is a non-commutative generalization of the usual (Euclidean) Yang–Mills action. Of course, as far as the calculation of the value of the Higgs mass is concerned, the metric independence of (19) does not matter. However, from a purely conceptual perspective the arbitrariness of the metric seems unsatisfying (like

in the usual Yang–Mills gauge theories) and may serve as the main motivation to change the definition of the bosonic action within non-commutative geometry from the Dixmier trace to the spectral action (cf [Con88, Con96]).

During the last decade different CLM approaches to the STM have been developed. Accordingly, there are also various statements about the predicted value of the mass of the Higgs within the frame of non-commutative geometry (see, for example, [Con95]). Within Connes' *real geometry* the fermionic representation, considered as an algebra representation, can be chosen differently from (29). As a consequence, the commutant is also differently parameterized. Moreover, one also obtains more freedom to parameterize the appropriate scalar products used to define the fermionic and bosonic actions in the (real) CLM. Note that the introduction of the real structure also yields a doubling of the gauge degrees of fermionic freedom quite similar to what is needed in order to introduce the Pauli–Dirac–Yukawa operator (21) (see also our discussion in [TT05]). Therefore, in contrast to the CCM and GTDT approach to the STM, within the (real) CLM one does not obtain a unique critical scale point  $t_c$  on which the corresponding parameter relations are assumed to hold true but, instead, a whole range of such points. This range corresponds to the energy interval

$$E_c \in [m_Z, 2 \times 10^5] \text{ GeV}. \quad (115)$$

Hence, in the (real) CLM the critical energy point  $E_c$  is at least by five orders less than in GTDT (and CCM). Accordingly, the predicted values of the Higgs mass are contained within the interval (again, for  $m_T = 175 \pm 6 \text{ GeV}$ )

$$m_H \in (194.5, 291] \text{ GeV}, \quad (\text{soft action}) \quad (116)$$

which has an empty intersection with (87) (cf [IKS95a, IKS95b, CIS99]). This, in fact, remains true even if one restricts the commutant and thus the parametrization in the CLM. For example, analogous to GTDT there is also a geometrically distinguished parametrization in (the real) CLM which gives rise to the following definite value of the Higgs mass on tree-level (for  $m_T = 175 \text{ GeV}$  and  $E_c = m_Z$ , cf loc. cit.):

$$m_H = 289 \text{ GeV}. \quad (117)$$

## 6. Conclusion

In this paper, we discussed the possible values of the Higgs mass as is predicted by the (minimal) Standard Model when the latter is considered as a specific gauge theory of Dirac type. We have shown that this approach to the STM permits us to yield (in a specific approximation including quantum corrections) a definite value of the Higgs mass without referring to additional assumptions coming, for instance, from cosmology. This is quite in contrast to the usual (non-geometrical) description of the STM, which only gives rise to a whole range of possible values of the Higgs mass. Within the GTDT approach to the STM the predicted value of the Higgs mass is in full accordance with the STM range, though it lies on the upper bound of the allowed interval. The presented approach of the STM clearly demonstrates once more the power of a geometrical understanding of physics and, in the case at hand, of the Standard Model of particle physics. This is emphasized by the circumstance, that a non-geometrical parametrization of a geometrical description may usually yield a ‘fuzziness’ of the predictive power. To demonstrate this we also discussed the most general (but non-geometrical) parametrization possible in the GTDT approach to the STM. Similar to the Connes–Lott approach to the STM this gives rise to an interval of possible values of the Higgs mass. In the frame presented, however, this interval has been shown to have

a non-empty intersection with the allowed STM range. In particular, the GTDT predicted lower bound is close to the lower bound of the STM range. In the presented geometrical scheme the fuzziness in the prediction of the value of the Higgs mass is originated only in the ‘self-interaction’  $[H_\mu, H_\nu]$  of the Higgs gauge potential. Of course, when expressed in terms of the usual Higgs potential  $V_H$  it is this self-interaction which gives rise to the mass of the physical Higgs boson.

We also discussed how the predicted value of the Higgs mass may depend on the existence of massive neutrinos. It turns out that the inclusion of Dirac-type mass terms to the STM does not alter the results presented. However, the inclusion of Majorana mass terms in our approach is less straightforward and not taken into account in this work but will be discussed separately in a forthcoming paper. Like in [CCM06] it is expected that such an inclusion in GTDT will also yield a (slightly) lower value of the Higgs mass predicted.

Since there are some similarities to other geometrical approaches to the STM, we included a brief comparison of our approach, in particular, with the Chamseddine–Connes and the (real) Connes–Lott approach to the STM concerning the Higgs mass.

Some comments on the role of gravity within GTDT may be worth mentioning. Actually, Einstein’s theory of gravity is an integral part of GTDT. Although neglected in our discussion of the Higgs mass, it plays a fundamental role in this approach to the STM. This is because it is intimately related to spontaneous symmetry breaking. Indeed, spontaneous symmetry breaking is considered as being due to the Higgs gauge potential

$$H_\mu \sim g_{\mu\nu} \gamma^\nu \gamma_5 \phi.$$

Accordingly, the role of the usual Higgs potential  $V_H$  is regarded as only giving rise to the mass of the Higgs boson. Concerning the numerical calculations of the value of the Higgs mass done in this paper, gravitational effects are assumed to be negligible (similar to other approaches). The physical reason that this can be done without contradictions within the setup of GTDT is that the two length-scales involved,  $\ell_H$  and  $\ell_P$ , are actually independent of each other. The drawback of this independence, of course, is that GTDT seems not to permit a unification of gravity with the strong and the electroweak interactions of the STM. On the other hand, on the energy scales considered one may not expect such a unification. In fact, one obtains for the critical energy point  $E_c$ , on which the parameter relations are shown to hold true, that  $E_c/m_P \lll 1$ . Accordingly, one has  $m_H/m_P \lll 1$  (resp.  $\ell_H/\ell_P \ggg 1$ ), as is usually expected (and also very much hoped for). At least, this demonstrates that the GTDT approach to the STM is consistent with the common assumption that gravity is generically negligible within the range of validity of the STM, although Einstein’s theory of gravity is naturally included within GTDT. This may also be inferred from the following rough qualitative considerations. The Euler–Lagrange equation of the Dirac Lagrangian (56) with respect to the metric yields the Einstein equation with the energy–momentum tensor being defined by  $\mathcal{L}_{STM}$ . When all field excitations are neglected (i.e. putting all fields equal to 0) this yields the non-vanishing scalar curvature

$$\begin{aligned} r_M &= -\frac{\pi}{2} \frac{\left(\frac{m_H}{m_P}\right)^2}{\lambda} m_H^2 \\ &\approx -\frac{10^{-13}}{2\pi} / \text{cm}^2. \end{aligned} \tag{118}$$

Here, we took into account the values of the Higgs mass (85) and of the Higgs self-coupling constant (83). Though already small value (118) should be contrasted with a typical cross section  $\sigma$  of a high energy process such that the dimensionless product  $r_M \sigma$  may be physically interpreted as the quotient of the two relative accelerations caused by gravitational



and high energy effects, respectively. Roughly, the order of magnitude of a typical  $\sigma$  ranges from  $\sigma \approx 10^{-35} \text{ cm}^2$  (weak interaction) to  $\sigma \approx 10^{-26} \text{ cm}^2$  (strong interaction). Hence,

$$|r_M \sigma| \lesssim 10^{-40} \quad (119)$$

which demonstrates again that gravitational effects are fully negligible on the scale  $\ell_H$ . Qualitatively, this does not change even if field excitations are taken into account, for example, by the replacement  $m_H \rightsquigarrow \kappa m_H$ , provided that  $1 \leq \kappa < \kappa_0 \equiv 10^{10}$ . Here,  $\kappa_0$  is defined such that  $|r_M \sigma| \approx 1$  where gravitational effects become comparable with typical high energy effects (of the strong interaction). Note that the critical energy scale  $E_c \approx 10^{10} \text{ GeV}$  corresponds to  $\kappa \approx 10^8$ . Thus, also on the critical energy scale where the parameter relations (61)–(64) are assumed to hold true, the high energy effects still significantly dominate the gravitational effects since  $|r_M \sigma| \lesssim 10^{-8}$ . This, however, does not hold true any longer for a critical energy scale  $E_c \approx 10^{13} - 10^{17} \text{ GeV}$  (cf [CIS99, KS06, CCM06]).

Irrespective of the concrete value and the geometrical scheme (commutative or non-commutative), it seems most remarkable that a prediction of a definite value of the Higgs mass can be obtained from the pure Standard Model without additional assumptions, provided the Standard Model is described in geometrical terms. Of course, that the Standard Model can be geometrically described at all is certainly in itself a quite remarkable fact, which one has to take into account in any theory that aims to go beyond the Standard Model.

### Appendix. Relations between gauge couplings and empirical Parameters

For the sake of convenience for the reader, in this appendix we recall how the gauge couplings  $g_1, g_2, g_3$  are related to the experimentally accessible parameters given by the strong-coupling constant  $\alpha_S$ , the fine structure constant  $\alpha$  and the electroweak mixing angle  $\sin^2 \vartheta_W$ . For the actual values of these empirical data we refer to [Y06a].

To obtain the relations between the gauge couplings and the above-mentioned empirical parameters we consider the electroweak sector of the fermionic Lagrangian density of the (minimal) Standard Model (here, the conventions used are those given in [Nac90]):

$$\mathcal{L}_{\text{elw}}^{\text{int}} = e \left\{ i A_\mu J_{\text{em}}^\mu + \frac{i}{\sin \vartheta_W \cos \vartheta_W} Z_\mu J_{\text{NC}}^\mu + \frac{i}{\sqrt{2} \sin \vartheta_W} (W_\mu^+ J_{\text{CC}}^\mu + W_\mu^- J_{\text{CC}}^{\mu\dagger}) \right\} d^4 x, \quad (\text{A.1})$$

with the currents being defined by

$$\begin{aligned} J_{\text{em}}^\mu &:= \bar{\psi} \gamma^\mu (T_3 + Y) \psi, \\ J_{\text{NC}}^\mu &:= \bar{\psi} \gamma^\mu (T_3 - \sin^2 \vartheta_W (T_3 + Y)) \psi, \\ J_{\text{CC}}^\mu &:= \bar{\psi} \gamma^\mu (T_1 + iT_2) \psi. \end{aligned} \quad (\text{A.2})$$

As usual, the physical fields are  $A_\mu, Z_\mu$  and  $W_\mu^\pm$ , where

$$\begin{aligned} A_\mu &:= \cos \vartheta_W W_\mu^3 - \sin \vartheta_W B_\mu, \\ Z_\mu &:= \sin \vartheta_W W_\mu^3 + \cos \vartheta_W B_\mu, \\ W_\mu^\pm &:= \frac{1}{\sqrt{2}} (W_\mu^1 \mp i W_\mu^2), \end{aligned} \quad (\text{A.3})$$

with, respectively,  $W_\mu^a$  ( $a = 1, 2, 3$ ) being the gauge fields of the  $SU(2)$  coupling and  $T_a$  are the appropriate generators. Here,  $B_\mu$  is the gauge field of the  $U(1)$  coupling with generator  $Y$  according to the fermionic representation. Note that, in contrast to the conventions used in the main text of the paper, the (non-physical) fields  $W_\mu^a$  and  $B_\mu$  are re-scaled:

$$W_\mu^a \rightarrow g_2 W_\mu^a, \quad B_\mu \rightarrow g_1 B_\mu. \quad (\text{A.4})$$

With these conventions in mind, the interaction Lagrangian density of the electroweak sector of the (minimal) Standard Model reads

$$\mathcal{L}_{\text{elw}}^{\text{int}} = \bar{\psi} i\gamma^\mu \left( \frac{e}{\cos \vartheta_W} B_\mu Y + \frac{e}{\sin \vartheta_W} W_\mu^a T_a \right) \psi d^4x. \quad (\text{A.5})$$

Accordingly, the gauge couplings are identified with

$$g_1 = \frac{e}{\cos \vartheta_W} = \frac{e}{\sqrt{1 - \sin^2 \vartheta_W}}, \quad g_2 = \frac{e}{\sin \vartheta_W}. \quad (\text{A.6})$$

Finally, using  $4\pi\alpha \equiv e^2$ , one gets

$$g_1 = \sqrt{\frac{4\pi\alpha}{1 - \sin^2 \vartheta_W}}, \quad g_2 = \sqrt{\frac{4\pi\alpha}{\sin^2 \vartheta_W}}. \quad (\text{A.7})$$

Similar to the definition of the fine structure constant, the relation between the  $SU(3)$ -coupling  $g_3$  and the strong-coupling constant  $\alpha_S$  reads

$$g_3 = \sqrt{4\pi\alpha_S}. \quad (\text{A.8})$$

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